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A Simple Positive Definite Advection Scheme with Small Implicit Diffusion

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ABSTRACT

The development of negative values for positive definite scalars in the solution of the advection equation is an important difficulty in numerical modeling. This paper proposes a new positively definite advection scheme which has a simple form, small implicit diffusion and low computational cost. Comparisons of the present scheme with some other known positive definite schemes are also presented.

1. Introduction

In numerical modeling of atmospheric phenomena it is often necessary to solve the advection equation for positive definite scalar functions. Using second-order or higher-order-accuracy advection schemes can introduce some difficulties because negative values arise in the solution (Soong and Ogura, 1973). This effect can be especially important in cases where the solution of the advection equation is used as input to nonlinear equations describing microphysical phenomena (e.g., the stochastic coalescence equation), which can eventually lead to instability of the whole system. Care about the positiveness of the solution leads to the use of upstream differencing or other low-order schemes (Soong and Ogura, 1973) which produce no dispersive "ripples" but which suffer from excessive numerical diffusion.

In the last ten years a possible resolution of this dilemma has been developed especially for application to numerical modeling of plasma fluid problems. The flux-corrected transport (FCT) method, developed by Boris and Book (1973, 1976), Book *et al.* (1975) and generalized for the multidimensional case by Zalesak (1979), and the self-adjusting hybrid scheme (SAHS) method developed by Harten (1978) and Harten and Zwas (1972) are based on a hybrid scheme in which the advective fluxes are given as a weighted average of the first-order positive definite scheme's fluxes and higher-order scheme's fluxes. Both methods were constructed to deal effectively with shocks and contact discontinuities. Solutions of the advection equation obtained by using FCT or SAHS are positive definite and as can be seen from presented tests (Zalesak, 1979; Harten, 1978) can be very accurate. Unfortunately, application of these methods to the modeling of atmospheric phenomena (especially multidimensional problems) is rather limited because of the excessive computer time required.

The idea of the hybrid scheme was applied to the atmospheric modeling by Clark (1979) and Clark and Hall (1979). They proposed a hybrid-type scheme based on a Crowley advection scheme (Crowley, 1968) as a higher-order scheme and "upstream" scheme as a low-order scheme. Although positive definiteness is not guaranteed in the scheme, the produced negative values are small enough to be neglected. The numerical diffusion in the scheme is larger than in, e.g., FCT but time consumption is about half that in FCT.

This paper presents another solution. Using an iterative approach, one can construct from the basis of the "upstream" scheme a new scheme which is positive defined but does not contain strong implicit diffusion. As will be shown later, the scheme is less time consuming than other positive-defined schemes and produces results which are comparable to the results obtained from the more complicated hybrid schemes.

In Section 2 the scheme and its development are presented. Section 3 contains the proof of the consistency and stability of the scheme. In Section 4 the results of comparison tests with the other positively-defined schemes are presented.

2. Development of a new positive definite advection scheme

The equation to be solved is the continuity equation describing the advection of a nondiffusive quantity in a flow field, i.e.,

$$\frac{\partial \psi}{\partial t} + \mathrm{div}(\mathbf{V}\psi) = 0, \tag{1}$$

where $\psi(x, y, z, t)$ is the nondiffusive scalar quantity,

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$\mathbf{V} = (u, v, w)$ is the velocity vector, and x, y, z, t are the space and time independent variables. For simplicity the one-dimensional case of (1),

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}(u\psi) = 0, \tag{2}$$

will be discussed. As will be shown later, the multidimensional case is a simple generalization of the one-dimensional results. As a base for construction of the proposed scheme the "upstream" advection scheme on a staggered grid was chosen:

$$\psi_i^{N+1} = \psi_i^N - \{F(\psi_i^N, \psi_{i+1}^N, u_{i+1/2}^N) - F(\psi_{i-1}^N, \psi_i^N, u_{i-1/2}^N)\}, \tag{3}$$

where

$$F(\psi_i, \psi_{i+1}, u) = [(u + |u|)\psi_i + (u - |u|)\psi_{i+1}]\frac{\Delta t}{2\Delta x}. \tag{4}$$

Here ψ_i^N is the value of ψ at the i grid point for N time step, Δt, Δx are the time and space increments, and the fluxes F are defined at the same staggered points as the velocity values.

The stability condition for scheme (3) has a form

$$\max_i\left(\frac{|u_{i+1/2}|\Delta t}{\Delta x}\right) \leqslant 1 \tag{5}$$

(in a case when velocity is time-dependent, Δt has to be adjusted at each time step or $\max\limits_i$ has to be replaced by $\max\limits_{i,N}$). Under condition (5), scheme (3) is positive definite which means:

$$(\psi_i^0 \geqslant 0 \quad \text{for all } i) \rightarrow (\psi_i^N \geqslant 0 \quad \text{for all } i \text{ and } N). \tag{6}$$

The properties (5) and (6) of scheme (3) as well as low computer time consumption are very useful for application of (3) to the numerical evaluation of (2). Unfortunately, scheme (3) is a first-order scheme (both in space and time) and has strong implicit diffusion. The rate of the implicit diffusion in (3) may be easily estimated for the case of uniform flow (u = const). Expanding ψ_i^{N+1}, ψ_{i+1}^N, ψ_{i-1}^N in a second-order Taylor sum about the point (x_i, t^N), scheme (3) may be written as

$$\left.\frac{\partial \psi}{\partial t}\right|_i^N = -\left.\frac{\partial}{\partial x}(u\psi)\right|_i^N + \frac{\partial}{\partial x}\left[0.5(|u|\Delta x - \Delta t u^2)\frac{\partial \psi}{\partial x}\right]\Bigg|_i^N. \tag{7}$$

From (7) it can be seen that scheme (3) approximates with second-order accuracy the equation

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}(u\psi) = \frac{\partial}{\partial x}\left(K_{\text{impl}}\frac{\partial \psi}{\partial x}\right), \tag{8}$$

where $K_{\text{impl}} = 0.5(|u|\Delta x - \Delta t u^2)$.

As Δt and $\Delta x \rightarrow 0$, Eq. (8) approaches Eq. (2) but during a realistic computational process scheme (3) with finite Δt and Δx approximates more accurately an advection equation with an additional diffusive term (8) rather than (2). On the other hand, this implicit diffusion term is important for stability of the scheme and cannot be simply subtracted from the scheme. An intuitively obvious approach is to make the advection step using (3) and then reverse the effect of the diffusion equation

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial x}\left(K_{\text{impl}}\frac{\partial \psi}{\partial x}\right) \tag{9}$$

in the next corrective step.

The problem is that the diffusion process and the equation that describes it are irreversible. But it is not true that the solution of the diffusion equation cannot be reversed in time. Just as a film showing the diffusion process may be reversed in time, the equivalent numerical trick may be found to produce the same effect. It is enough to notice that (9) may be written in the form

$$\frac{\partial \psi}{\partial t} = -\frac{\partial}{\partial x}(u_d\psi), \tag{10}$$

where

$$u_d \equiv \begin{cases} -\dfrac{K_{\text{impl}}}{\psi}\dfrac{\partial \psi}{\partial x}, & \text{if} \quad \psi > 0 \\ 0, & \text{if} \quad \psi = 0. \end{cases} \tag{11}$$

Here u_d will be referred to later as the "diffusion velocity." Now, defining an "antidiffusion velocity"

$$\tilde{u} = \begin{cases} -u_d, & \text{if} \quad \psi > 0 \\ 0, & \text{if} \quad \psi = 0, \end{cases} \tag{12}$$

the reversal in time of the diffusion equation (9) may be simulated by the advection equation (10) with an "antidiffusion velocity" $\tilde{u}$. Based on these concepts, the following advection scheme is suggested:

1) $$\psi_i^* = \psi_i^N - \{F(\psi_i^N, \psi_{i+1}^N, u_{i+1/2}^N) - F(\psi_{i-1}^N, \psi_i^N, u_{i-1/2}^N)\}, \tag{13}$$

2) $$\psi_i^{N+1} = \psi_i^* - \{F(\psi_i^*, \psi_{i+1}^*, \tilde{u}_{i+1/2}) - F(\psi_{i-1}^*, \psi_i^*, \tilde{u}_{i-1/2})\}, \tag{14}$$

where

$$\tilde{u}_{i+1/2} = \frac{(|u_{i+1/2}|\Delta x - \Delta t u_{i+1/2}^2)(\psi_{i+1}^* - \psi_i^*)}{(\psi_i^* + \psi_{i+1}^* + \epsilon)\Delta x}, \tag{15}$$

where F is defined as in (4), and ϵ is a small value, e.g., 10^{-15}, to ensure $\tilde{u} = 0$ when $\psi_{i+1}^* = \psi_i^* = 0$. It is assumed that $\psi_i^0 \geqslant 0$, for all i. The proposed scheme may be written in one step only, but this is only a matter of efficient coding.

3. Stability and consistency of proposed scheme

The proposed scheme is constructed from the well known, stable and consistent "upstream" scheme. To show consistency of the whole scheme [(13), (14)] it is enough to show that when Δx, $\Delta t \rightarrow 0$, the second step of scheme 2) does not affect the solution of the first step 1). Because of the stability condition (5) for the "upstream" scheme and the fact that as $\Delta x \rightarrow 0$, $\psi_{i+1} - \psi_i \rightarrow 0$, it is easy to see that $\tilde{u} \rightarrow 0$ in (15). So, using (4), (14), (15) and dividing both sides of (14) by Δt, Eq. (14) approaches the form

$$\frac{\partial \psi}{\partial t} = 0, \tag{16}$$

which means that the scheme (13), (14) is consistent.

To show stability of the scheme it is enough to show that stability of the step 1) implies the stability of 2). Because step 2) is also an upstream scheme, stability condition (5) takes the form

$$\max_i \left(\frac{|\tilde{u}_{i+1/2}| \Delta t}{\Delta x} \right) \leqslant 1. \tag{17}$$

Using Eq. (15) condition (17) may be written as

$$\max_i \left(\underbrace{\frac{|u_{i+1/2}|\Delta t}{\Delta x}}_{\text{A}} \cdot \underbrace{\left(1 - \frac{|u_{i+1/2}|\Delta t}{\Delta x}\right)}_{\text{B}} \times \underbrace{\frac{|\psi^*_{i+1} - \psi^*_i|}{(\psi_{i+1} + \psi^*_i + \epsilon)}}_{\text{C}} \right) \leqslant 1. \tag{18}$$

The terms A and B are less than or equal to unity because of (5). Term C is less than or equal to unity because ψ^* was obtained from the positive definite "upstream" scheme 1). It is important to notice that the maximum of $\text{A} \cdot \text{B} = 0.25$ which allows values at $\tilde{u}$ to increase without destroying the stability of the scheme. In a multidimensional case the scheme may be used in the time-splitting or the combined form, optionally. [The differences between time-splitting and combined schemes are discussed in detail in Smolarkiewicz (1982).] In the time-splitting form the stability and consistency of the scheme is a consequence of the stability and consistency of the one-dimensional scheme.

When the scheme is applied in combined form (e.g., for the two-dimensional case), Eqs. (13) and (14) may be written as

$$\begin{aligned}\psi^*_{ij} = \psi^N_{ij} - \{&F(\psi^N_{ij}, \psi^N_{i+1,j}, u^N_{i+1/2,j}) \\ &- F(\psi^N_{i-1,j}, \psi^N_{ij}, u^N_{i-1/2,j}) + F(\psi^N_{ij}, \psi^N_{i,j+1}, v^N_{i,j+1/2}) \\ &- F(\psi^N_{i,j-1}, \psi^N_{ij}, v^N_{i,j-1/2})\},\end{aligned} \tag{19}$$

$$\begin{aligned}\psi^{N+1}_{ij} = \psi^*_{ij} - \{&F(\psi^*_{ij}, \psi^*_{i+1,j}, \tilde{u}_{i+1/2,j}) \\ &- F(\psi^*_{i-1,j}, \psi^*_{ij}, \tilde{u}_{i-1/2,j}) + F(\psi^*_{ij}, \psi^*_{i,j+1}, \tilde{v}_{i,j+1/2}) \\ &- F(\psi^*_{i,j-1}, \psi^*_{ij}, \tilde{v}_{i,j-1/2})\},\end{aligned} \tag{20}$$

where $\tilde{v}$ is an expression as in (15). In this case, a sufficient stability condition for the upstream scheme is

$$\max_{i,j} \left(\frac{u^2_{i+1/2,j}\Delta t^2}{\Delta x^2} + \frac{v^2_{i,j+1/2}\Delta t^2}{\Delta y^2} \right)^{1/2} \leqslant 2^{-1/2} \tag{21}$$

[cf. Eq. (3-140) of Roache (1972), which implies (21)] and for the second step of scheme (20)

$$\max_{i,j} \left(\frac{\tilde{u}^2_{i+1/2,j}\Delta t^2}{\Delta x^2} + \frac{\tilde{v}^2_{i,j+1/2}\Delta t^2}{\Delta y^2} \right)^{1/2} \leqslant 2^{-1/2}. \tag{22}$$

As in (18), it is easy to show that if (21) is valid then the left-hand side of (22) is equal to $2^{1/2}/4$, so (22) is valid too. In the three-dimensional case the expressions equivalent to (21) and (22) have left-hand sides whose maxima are $3^{-1/2}$ and $3^{1/2}/4$, respectively. So, one can conclude that the "antidiffusive velocities" can be increased by factors of 4, 2, $4/3$, respectively, for one-, two- and three-dimensional cases, without destroying the stability of the scheme. This property of the scheme is very useful. Because the corrective step is an "upstream" scheme, it also introduces some implicit diffusion. Certainly, the procedure could be repeated but that would double the time consumption of the scheme. Additionally, in the multidimensional combined case, the proposed procedure does not compensate the implicit, cross partial derivative terms, so repetition of the corrective step of the scheme would amplify the cross-term effect. [The role of the cross-term in advection scheme was discussed in Smolarkiewicz (1982).] The compensation of the implicit diffusion in the second step of the scheme can be done by increasing "antidiffusion velocities" by some factor *Sc*, i.e.,

$$\tilde{u}_{\text{new}} = Sc\tilde{u}. \tag{23}$$

As will be shown in the next section, this factor can be estimated experimentally and significantly improves the quality of the solution.

4. The results of the tests

To compare the behavior of the proposed scheme with the behavior of the FCT, SAHS and Clark and Hall's schemes the two-dimensional solid body rotation test was chosen. The grid space was ($100\Delta x$ by $100\Delta y$), $\Delta x = \Delta y = 1$, $\Delta t = 0.1$, and the constant angular velocity $\omega = 0.1$. The velocity components are $u = -\omega(y - y_0)$ and $v = \omega(x - x_0)$ where $(x_0, y_0) = (50\Delta x, 50\Delta y)$. In this circumstance, the maximum value of the Courant number $(u^2\Delta t^2/\Delta x^2 + v^2\Delta t^2/\Delta y^2)^{1/2}$ was 0.7, and one full rotation around the point

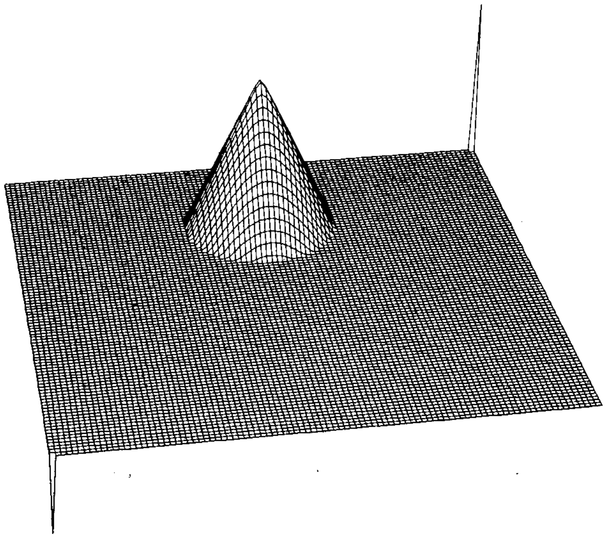

FIG. 1. Initial condition for all tests. Scale values in left-front and right-back corners are −2 and 4, respectively. The scale values are the same in all figures.

(x_0, y_0) was equivalent to 628 iterations (i.e., time steps). The initial condition was assumed in a form of a cone with base radius $15\Delta x = 15\Delta y$ and maximum value 3.87 in point $(X_m, Y_m) = (75\Delta x, 50\Delta y)$ (Fig. 1). In all cases the same boundary conditions were used. The first spatial partial derivative in the normal direction was assumed to vanish at the outflow boundary (vanishing of the second derivative does not insure the positive definition of the proposed scheme). The undisturbed initial value of the field was assumed at the inflow boundary. The hybrid schemes, with which the proposed scheme has been compared, can be written for the two-dimensional case in general form

$$\psi_{ij}^{N+1} = \psi_{ij}^{N} - \{[\mathbf{CX}\cdot \mathrm{FHX} + (1 - \mathbf{CX})\cdot \mathrm{FLX}]_{i+1/2,j} - [\mathbf{CX}\cdot \mathrm{FHX} - (1 - \mathbf{CX})\cdot \mathrm{FLX}]_{i-1/2,j} + [\mathbf{CY}\cdot \mathrm{FHY} + (1 - \mathbf{CY})\cdot \mathrm{FLY}]_{i,j+1/2} - [\mathbf{CY}\cdot \mathrm{FHY} - (1 - \mathbf{CY})\cdot \mathrm{FLY}]_{i,j-1/2}\}, \quad (24)$$

where FHX, FHY are the advection fluxes in orthogonal directions from a high-order-accurate advection scheme (second order or above); FLX, FLY are the orthogonal direction fluxes from a first-order positive definite advection scheme; and **CX**, **CY** are the matrices of corrective factors $[CX_{i+1/2,j}]$, $[CY_{i,j+1/2}]$, where the value of each factor is less than or equal to unity and larger than or equal to zero $(0 \leqslant C \leqslant 1)$.

For all tests the FHX, FHY fluxes were assumed to be from the Crowley advection scheme (Crowley, 1968), i.e.,

$$\mathrm{FHX}_{i+1/2,j} = \frac{u_{i+1/2,j}\Delta t}{2\Delta x}(\psi_{i+1,j} + \psi_{ij}) - \frac{1}{2}\left(\frac{u_{i+1/2,j}\Delta t}{\Delta x}\right)^2(\psi_{i+1,j} - \psi_{ij}), \quad (25)$$

and FLX, FLY were assumed the "upstream" scheme fluxes of (4).

The major differences between the tested schemes (FCT, SAHS and Clark and Hall's method) are in the determination of the corrective factors CX, CY. In the case of the FCT method, scheme CX, CY were determined following Zalesak. [1979, Section VI, formulas 6′ to 13′ and 17′, 18′; in my experience, the

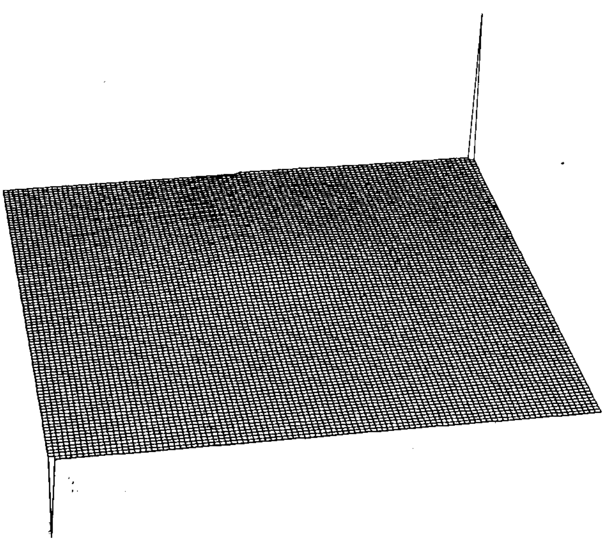

FIG. 2. Solution for "upstream" scheme after six full rotations (3768 iterations).

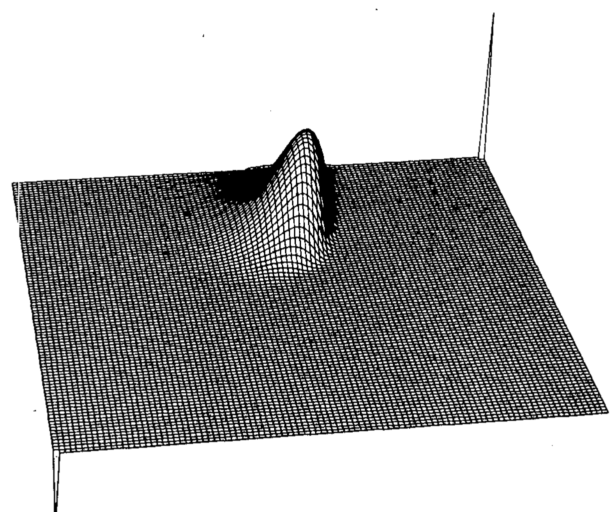

FIG. 3. As in Fig. 2, but for Clark-Hall hybrid schemes.

optional use of formula 14′ (or 14 in the one-dimensional case) gives very accurate solutions for an initial condition which is some kind of step function, but in a case of a smooth initial condition still tends to convert the solution to a step function.]

In the case of SAHS scheme, CX, CY were determined following Harten. [1978, Section 4, p. 371, formulas 4.12 a, 4.12 b with $p = 1$, $\epsilon = 10^{-15}$, $\xi(w) = w$ (notation as in referenced paper), CX in (24) is dependent on CX defined by Harten by relation $CX = 1 - CX_{\text{Harten}}$.] In the tested case only this part of the SAHS scheme has been applied which insures positive definiteness of the scheme (Harten, 1978, theorem, p. 372) while the optional "artificial compressor" has been omitted. Such treatment of SAHS was chosen because the full SAHS is more similar to the FCT (in complication level, time consumption and quality of results) than to the Clark-Hall or proposed scheme. In contrast to the complicated form of CX, CY for FCT or SAHS, corrective factors have very simple forms in the case of the Clark-Hall scheme. Following Clark (1979) CX may be written as

$$CX_{i+1/2,j} = \begin{cases} 1 - \left(\dfrac{\psi_{i+i,j} - \psi_{ij}}{|\psi_{i+1,j}| + |\psi_{ij}| + \epsilon}\right)^2, & \text{if } CX_{i+1/2,j} \leq 1/2 \\ 1, & \text{if } CX_{i+1/2,j} > 1/2 \end{cases} \quad \epsilon = 10^{-10}. \tag{26}$$

The form of (26) was postulated intuitively and finally determined experimentally (Clark, Hall, personal communications, 1982) and the role of the threshold value 1/2 is still unclear. I have found experimentally that even better final results may be obtained when 1/2 in (26) is replaced by the Courant number $u_{i+1/2,j}\Delta t/\Delta x$. The latter is consistent with information (Clark, Hall, personal communication, 1982) that most calculations done with (26) were in situations where the Courant number was $\sim 1/2$. Although (26) does not ensure positive definiteness of scheme (24), during all calculations the lowest negative values obtained were of order 10^{-9}–10^{-13}. The three different versions of the hybrid scheme presented above were compared with the following versions of the present scheme: (i) six versions of [(19), (20)] with $Sc = 1$, 1.02, 1.04, 1.06, 1.08 and 1.1; (ii) [(19), $(20)^2$] which means that corrective step (20) was applied

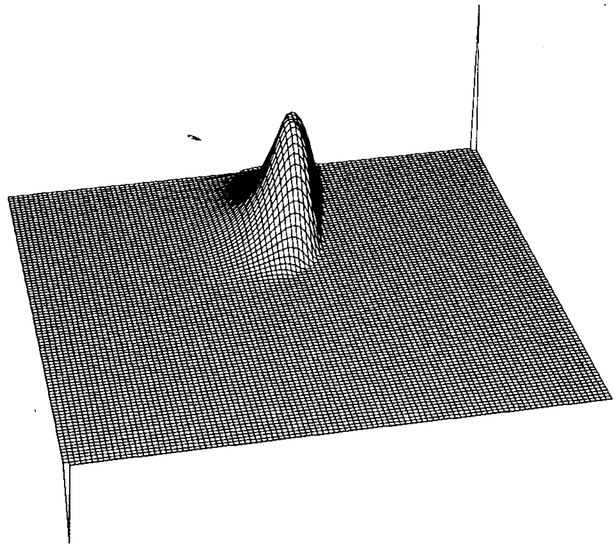

FIG. 5. As in Fig. 2, but for FCT hybrid scheme.

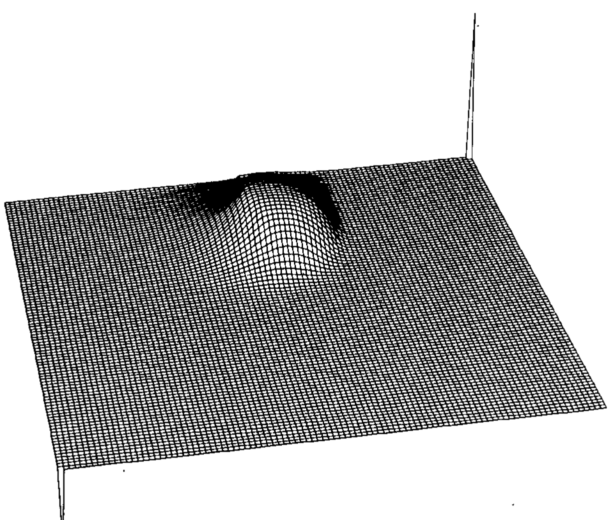

FIG. 4. As in Fig. 2, but for SAHS hybrid scheme.

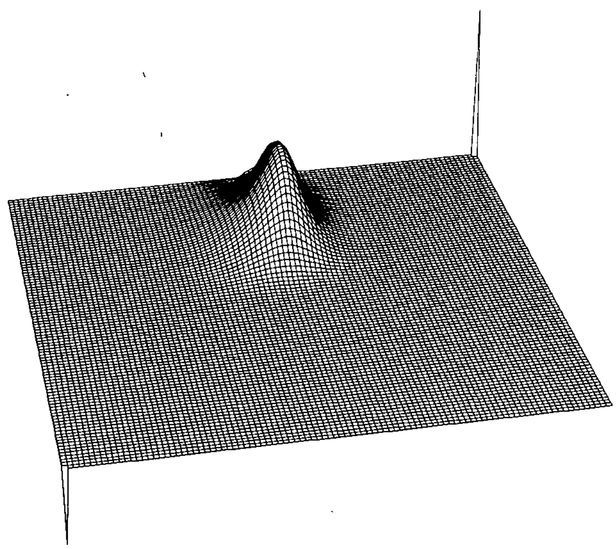

FIG. 6. As in Fig. 2, for basic version of proposed scheme, $[(19), (20)]Sc = 1$.

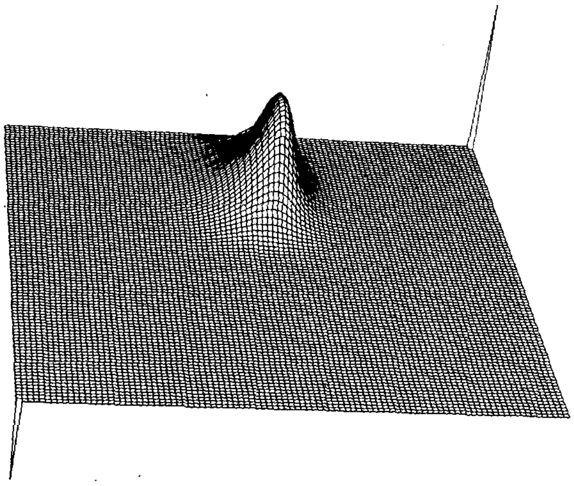

FIG. 7. As in Fig. 6, but for $Sc = 1.02$.

twice, with $\tilde{\tilde{u}} = \tilde{u}(\tilde{u}, \psi^{**})$ in the third step and ψ^{**} instead of ψ^{N+1} in the second step; (iii) $[(13), (14)]_{TS}$, $[(13), (14)^2]_{TS}$, $[(13), (14)^3]_{TS}$ where the subscript TS means that the scheme was applied in "time-splitting" form. For all tested schemes the values characterizing the scheme are shown after 3768 iterations (6 full rotations) in Table 1. The meanings of the applied abbreviations are as follows:

MAX (maximum value after six rotations)/(maximum value at initial time),

MIN minimum value after six rotations (initial minimum value = 0)

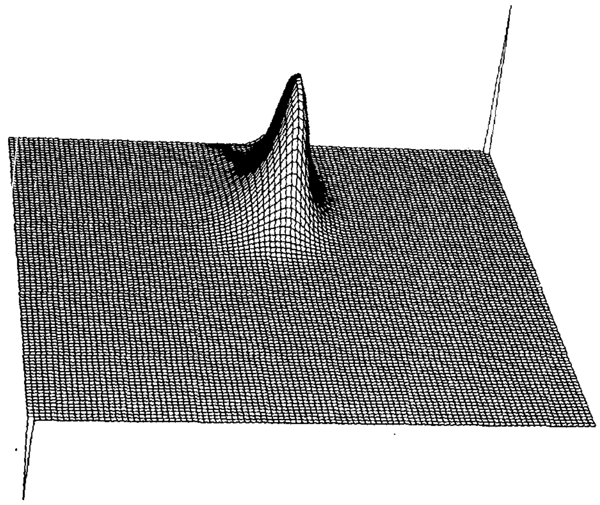

FIG. 9. As in Fig. 6, but for $Sc = 1.06$.

TC (computer time consumption)/(computer time consumption for "upstream" scheme)

ER2 error of the conservation of ψ^2

$$ER2 = 1 - \left\{\iint_\Sigma \psi^2(x, y, t)dxdy + \int_0^t [\text{outflow}(\psi^2)]dt\right\} \times \left\{\iint_\Sigma \psi^2(x, y, 0)dxdy\right\}^{-1},$$

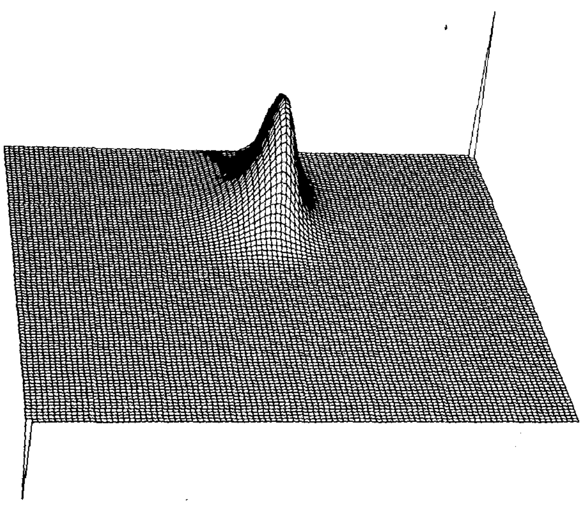

FIG. 8. As in Fig. 6, but for $Sc = 1.04$.

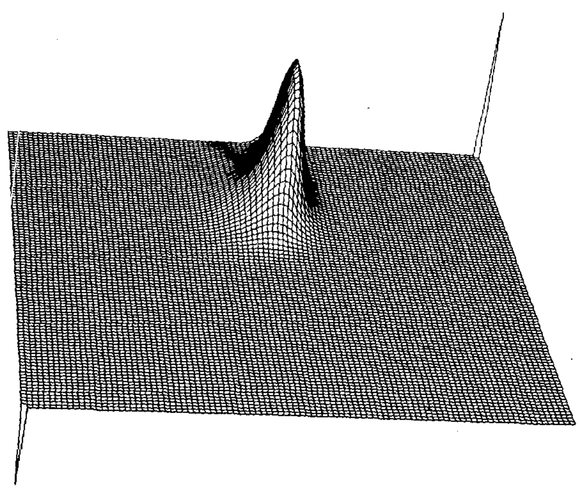

FIG. 10. As in Fig. 6, but for $Sc = 1.08$.

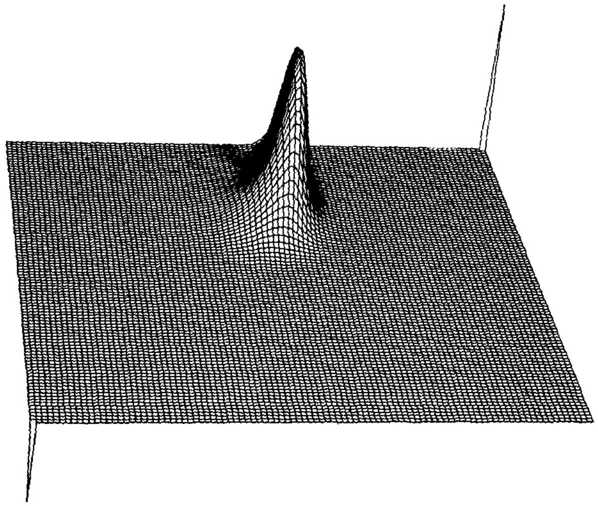

FIG. 11. As in Fig. 6, but for $Sc = 1.01$.

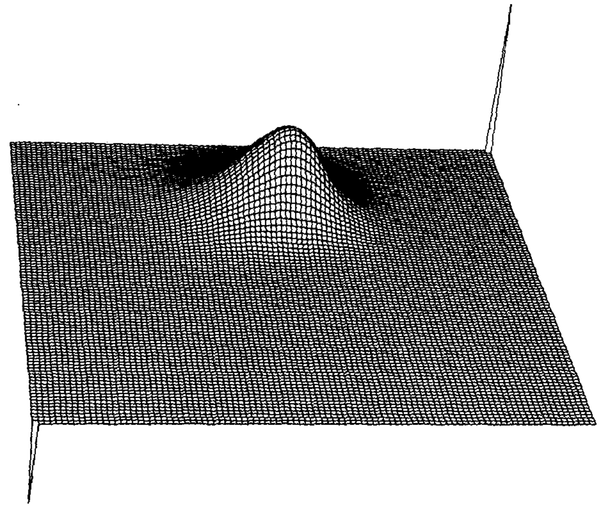

FIG. 13. As in Fig. 6, but for time splitting version of scheme, $[(13), (14)]_{TS}$.

where Σ is the whole domain x, y, and the column labeled "Fig." in Table 1 indicates the number of the figure on which results are presented. The value of the error of the conservation of ψ is not shown in Table 1. All schemes are in conservative flux form and for all cases ER1 was $\sim 10^{-12}$, where ER1 was defined as

$$\mathrm{ER1} = 1 - \left\{\iint_{\Sigma} \psi(x, y, t)dxdy + \int_0^t [\mathrm{outflow}(\psi)]dt\right\} \times \left\{\iint_{\Sigma} \psi(x, y, 0)dxdy\right\}^{-1}.$$

Analyzing values presented in Table 1 the following conclusions can be formulated:

1) In a multidimensional case the corrective step should not be repeated because it leads to increased "energy" in the whole system $[(19), (20)^2]$, Fig. 12. This is because in the multidimensional case the implicit cross-space partial derivative terms are not compensated. Repetition of the corrective step amplified this effect. When the scheme is applied in "time-splitting" form implicit cross terms are compensated and corrective steps may be repeated optionally $\{[13, 14]_{TS}, [13, (14)^2]_{TS}, [13, (14)^3]_{TS}$, Figs. 13, 14, 15$\}$.

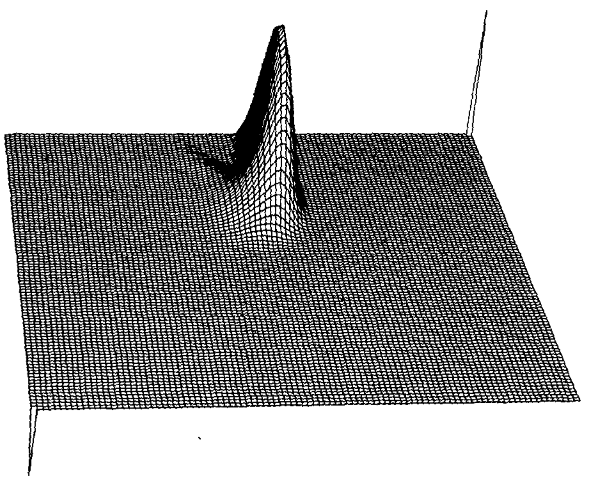

FIG. 12. As in Fig. 6, but for version of scheme with repeated corrective step, $[(19), (20)^2]$.

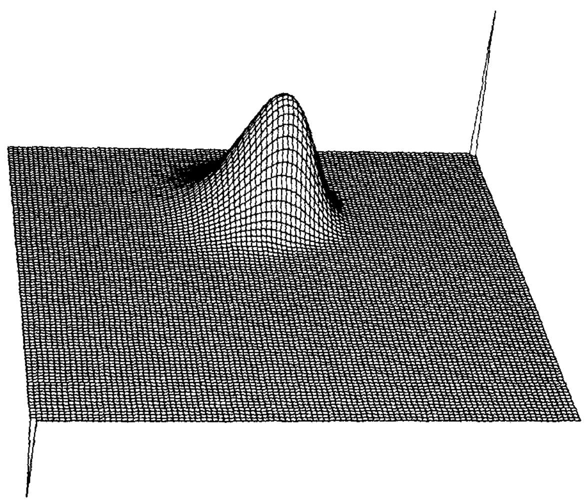

FIG. 14. As in Fig. 13, but with repeated corrective step, $[(13), (14)^2]_{TS}$.

From comparison of Figs. 14 and 15 it can be seen that more than one repetition of the corrective step is rather inefficient, because it improves accuracy only slightly but increases the time consumption of the scheme. The $[(13), (14)^2]_{TS}$ version of the scheme gives better results than any tested hybrid type scheme.

2) For practical application, in a case where the low time consumption of the scheme is required, use of the [(19), (20)] version of the scheme with a coefficient Sc seems to be the best option. Using $Sc = 1.06$ (Fig. 9) gives better results than any hybrid scheme. For the considered tests the range of optimal values of coefficient Sc was obtained $1 \leqslant Sc \leqslant 1.08$ ($Sc = 1.1$ increases maximum value over initial value). Probably for other flow fields this range may be slightly different, but always can be found using some simple tests.

TABLE 1. Comparison between different positive defined schemes for solid body rotation test.

Scheme	MAX	MIN	TC	ER2	Fig.
Upstream	0.07	0.	1.	0.95	2
Clark, Hall	0.65	-5×10^{-10}	3.7	0.39	3
SAHS	0.37	-5×10^{-23}	3.9	0.63	4
FCT	0.79	-1×10^{-15}	7.7	0.29	5
[(19), (20)]					
$Sc = 1$	0.60	0.	2.9	0.52	6
$Sc = 1.02$	0.68	0.	2.9	0.46	7
$Sc = 1.04$	0.77	0.	2.9	0.39	8
$Sc = 1.06$	0.85	0.	2.9	0.31	9
$Sc = 1.08$	0.96	0.	2.9	0.24	10
$Sc = 1.1$	1.20	0.	2.9	0.16	11
$[(19), (20)^2]$	1.26	0.	5.0	−0.10	12
$[(13), (14)]_{TS}$	0.56	0.	3.2	0.51	13
$[(13), (14)^2]_{TS}$	0.81	0.	5.2	0.19	14
$[(13), (14)^3]_{TS}$	0.84	0.	7.2	0.13	15

5. General conclusion

1) Using the "upstream" advection scheme and reducing the implicit diffusion by using a second "upstream" step with a specially defined velocity field leads to a new form of a positive definite advection scheme with small implicit diffusion. The obtained scheme has a simple form that is computationally efficient.

2) Implicit diffusion of the second corrective step of the scheme may be reduced by introducing a "correction coefficient" $Sc(1 \leqslant Sc \leqslant 1.08)$ which significantly improves the results obtained.

3) When the proposed scheme is applied in "time-splitting" form, the corrective step of the scheme may be repeated optionally. This gives the possibility of obtaining a more accurate solution. The practical application of even one repetition of the corrective step gives results more accurate than any tested hybrid scheme.

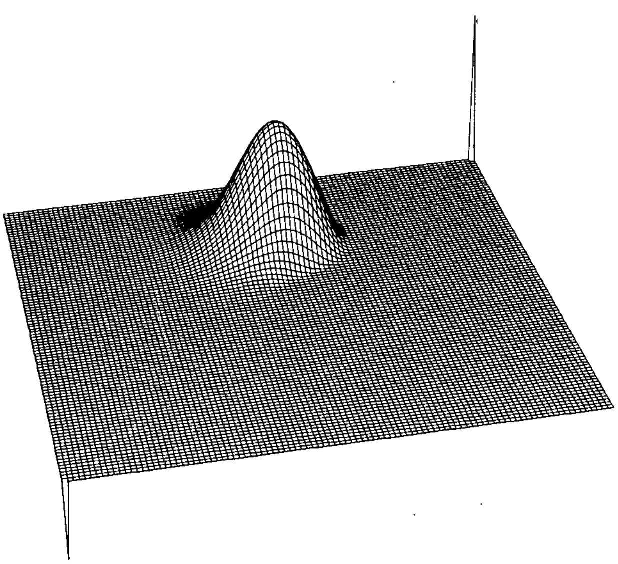

FIG. 15. As in Fig. 13, but with twice repeated corrective step, $[(13), (14)^3]_{TS}$.

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